Faculty of science **Third year**

Department of Mathematics **Numbers theory**

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The perfect answer :

(1) If a, b are integers numbers, the number m is said to be the least common multible of a, b and written [a, b] if

i) $m > 0$ ii) $a \mid m$, $b \mid m$ iii) if $a \mid c$ and $b \mid c$, then $m \le c$.

Also, $[6, 15, 20] = 30$

(2) Let $(a , b) = d$.

$\Rightarrow d \mid a$ and $d \mid b$

$\Rightarrow d \mid aq$ and $d \mid b$

$\Rightarrow d \mid (b - aq) = r$

$\Rightarrow d \mid a$, $d \mid r$

$\Rightarrow d \le (a , r)$

$\Rightarrow (a , b) \le (a , r)$ (1)

Conversely , let $(a , r) = c$

$\Rightarrow c \mid a$ and $c \mid r$

$\Rightarrow c \mid aq$ and $c \mid r$

$\Rightarrow c \mid aq + r = b$

$\Rightarrow c \mid a$, $c \mid b$

$\Rightarrow c \le (a , b)$

$\Rightarrow (a , r) \le (a , b)$ (2)

(1),(2) $\Rightarrow (a , r) = (a , b)$.

(3) Let $(a, b) = d \mid c$.

$\Rightarrow d \mid a \quad , \quad d \mid b \quad \text{and} \quad d \mid c$

$\Rightarrow \exists\ k, s, t$ such that $a = k\,d$, $b = s\,d$ and $c = t\,d$

$\Rightarrow d = a/k$, $d = b/s$ and $c = t\,d$

$\Rightarrow d = (1/2k)\,a + (1/2s)\,b$ and $c = t\,d$

$\Rightarrow d = m\,a + n\,b$ and $c = t\,d$

$\Rightarrow c = t\,d = t\,m\,a + t\,n\,b$

By comparison with the given linear equation we have

$x = t\,m$ and $y = t\,n$ is the solution of the equation $a\,x + b\,y = c$

Conversely , let x_o , y_o is the solution of the equation $a\,x + b\,y = c$

$\Rightarrow \quad a\,x_o + b\,y_o = c$

since $(a, b) = d \quad \Rightarrow \quad d \mid a$ and $d \mid b$

$\Rightarrow \quad d \mid a\,x_o + b\,y_o$

$\Rightarrow \quad d \mid c$.

(4) $360 = 2 \times 123 + 114$;

$123 = 1 \times 114 + 9$;

$114 = 12 \times 9 + 6$;

$9 = 1 \times 6 + 3$;

$6 = 2 \times 3 + 0$.

$\Rightarrow (360 , 123) = 3 \mid 99$.

So, the given equation has a solution

$3 = 9 - 6$;

$= 9 - 114 + 12 \times 9$

$= 9 \times 13 - 114$

$= 13 \times (123 - 114)$

$= 13 \times 123 - 13\,(360 - 2 \times 123)$

$= 39 \times 123 - 13 \times 360$

$\Rightarrow 99 = (33 \times 39)\,123 - (33 \times 13)\,360$

$= 1287 \times 123 - 429 \times 360$

$= 123\,x - 360\,y$

$\Rightarrow\ x_o = 1287 \quad ; \qquad y_o = -429 .$

$\Rightarrow\ x = x_o + k\,b/d = 1287 + 120\,k \quad ; \quad y = y_o - k\,a/d = -429 - 41\,k .$

(5) If $a \equiv b \pmod n$ and $c \equiv e \pmod n$, then

$n \mid a - b$ and $n \mid c - e$

$\Rightarrow\ \exists\ k, s$ such that $a - b = k\,n$ and $c - e = s\,n$

$\Rightarrow\ (a - b) - (c - e) = (k - s)\,n$

$\Rightarrow\ n \mid (a - b) - (c - e) = (a - c) - (b - e)$

$\Rightarrow\ a - c \equiv b - e \pmod n$ and

$a\,.\,c - b\,.\,e = a\,c + b\,c - b\,c - b\,e = c\,(a - b) + b\,(c - e) = c\,k\,n + b\,g\,n$

$= (c\,k + b\,g)\,n$

$= ln$

$\Rightarrow\ n \mid a\,c - b\,e$

$\Rightarrow\ a\,.\,c \equiv b\,.\,e \pmod n.$

(6) The Euler function $\varphi(n)$ is numbers which are relatively prime with n.

$\varphi(720) = \varphi(2^4 \times 3^2 \times 5) = 720\,(1 - 1/2).\,(1 - 1/3).\,(1 - 1/5) = 192$

Also, let n be a prime number , so 1, 2, 3 , …, n-1 are prime with n

Thus $\varphi(n) = n - 1$.

Conversely, let $\varphi(n) = n - 1$.

if n is not prime number, there is d which devisor of n such that

$1 < d < n$ and $(n , d) = d$.

i . e . there is at least one number of 1, 2, …, n –say d- not relatively prime with n.

$\therefore\ \varphi(n) \le n - 2$. This is a contradiction, therefore n must be a prime.

(7) The functions σ , τ are not perfect multiplicative because.

$$\sigma(2 \times 4) = \sigma(8) = 15 \neq 3 \times 7 = \sigma(2) \times \sigma(4)$$

$$\tau(2 \times 4) = \tau(8) = 4 \neq 2 \times 3 = \tau(2) \times \tau(4)$$

Also, $\sigma(228) = 560$ and $\tau(100) = 9$.

(8) The positive integers x, y, z are called primitive Phythagorean triple if

$x^2 + y^2 = z^2$ and $(x, y, z) = 1$

Let x , y , z are Phythagorean triple, then $x^2 + y^2 = z^2$ and $(x, y, z) = d$

Then $d \mid x$, $d \mid y$, $d \mid z$

$\Rightarrow \exists\, x_1 , y_1$, z_1 such that $x = x_1 d$, $y = y_1 d$, $z = z_1 d$.

Since $x^2 + y^2 = z^2$, then $x_1^2 d^2 + y_1^2 d^2 = z_1^2 d^2$. That is $x_1^2 + y_1^2 = z_1^2$

and $(x_1, y_1, z_1) = 1$

Thus x_1, y_1, z_1 are primitive Phythagorean triple.

The inverse is true because if x_1, y_1, z_1 are primitive Phythagorean triple, then is

$x_1^2 + y_1^2 = z_1^2$

By multiplicative of d^2 we have $x_1^2 d^2 + y_1^2 d^2 = z_1^2 d^2$. That $x^2 + y^2 = z^2$

Hence x , y , z are Phythagorean triple

To prove that $(x_1 , y_1) = 1$.

Let $(x_1 , y_1) = d > 1$

$\Rightarrow d \mid x_1$ and $d \mid y_1$ (1)

$\Rightarrow d \mid x_1^2$ and $d \mid y_1^2$

$\Rightarrow d \mid x_1^2 + y_1^2$

$\Rightarrow d \mid z_1^2$

$\Rightarrow d \mid z_1$ (2)

(1),(2) $\Rightarrow$ $(x_1, y_1, z_1) > 1$ which is a contradiction.