جامعة بنها   دراسات عليا (تمهيدي ماجستير)
كلية العلوم   مادة (إحصائية متقدمة)
دور يناير 2016   الزمن 3 ساعات

د./ صلاح عيد إبراهيم حمزة

ورقة كاملة شعبة الفيزياء الجوامد

1. Discus in details the black body radiation phenomena

------------------------------------- **Solution** -------------------------------------

A blackbody refers to an opaque object that emits thermal radiation. A perfect blackbody is one that absorbs all incoming light. If heated to a high temperature, a blackbody will begin to glow with thermal radiation. An approximation to a perfect blackbody may be obtained by a hollow sphere having a small opening with the inside walls having a rough, dull surface as shown in Fig. (11). The radiation enters or leaves the cavity through a small hole. Part of the radiation entering the cavity will be absorbed by its walls and part reflected. Only a very small part of the reflected radiations escape through the hole, so that after many internal reflections nearly all the radiation is absorbed and the body approximates a blackbody.

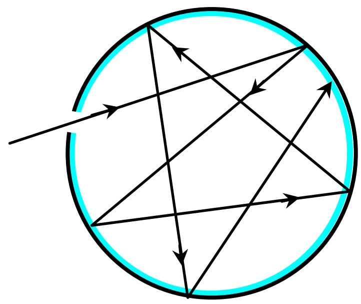


Fig. (11): An approximation of a perfect absorber (Blackbody)

At the beginning of the 20th century, scientists Lord Rayleigh, an Max Plank studied the black body radiation using such a device. After much work, Planck was able to describe the intensity of light emitted by a blackbody as a function of wavelength. **Planck's work on blackbody is one the areas of physics that led to the foundation of the wonderful**

science of Quantum Mechanics, but that is unfortunately beyond the scope of this course.

Planck was found that the temperature of a blackbody increases, the total amount of light emitted per second increases, and the wavelength of the spectrum's peak shifts to the bluer colors, see Fig. (12). For example, an iron bar becomes orange-red when heated to high temperatures and its color shifts toward blue and white as it is heated further.

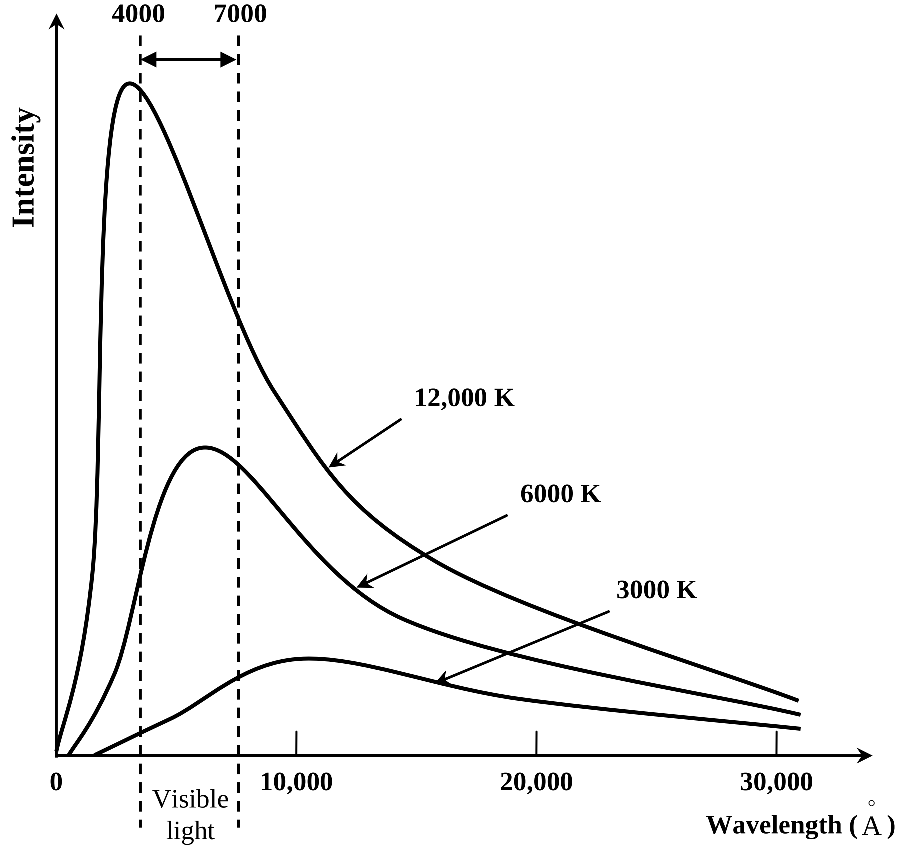


Fig. (12): The wavelength of the spectrum's peak shifts to the bluer colors

The Black body radiations can be considered as the photon gas. Photons are taken as bosons and they are obey BE statistics

$$n_i = \frac{g_i}{e^{\varepsilon_i / KT} - 1} \qquad (1)$$

We can write

$$dn = \frac{g(\nu)d\nu}{e^{h\nu lKT} - 1} \tag{2}$$

The translational kinetic energy for a particle in a cubical box is

$$\varepsilon = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) \tag{3}$$

In gama space

$$r^2 = n_x^2 + n_y^2 + n_z^2 \tag{4}$$

$$r^2 = \frac{8mL^2}{h^2}\varepsilon \tag{5}$$

So

$$gd\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon \tag{6}$$

Since the photon has no rest mass, so we can write

$$gd\nu = \frac{4\pi V}{h^3}\frac{h^2\nu^2}{c^2}\frac{h}{c}d\nu \tag{7}$$

The energy per unit volume, energy density, is

$$\rho d\nu = \frac{dn}{V}h\nu \tag{8}$$

So

$$\rho d\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/KT} - 1}d\nu \tag{9}$$

Which represents Planck's radiation law

Wien's Law

Wilhelm Wien quantified the relationship between blackbody temperature and the wavelength of the spectral peak with the following equation:

$$\lambda_{max} \cdot T = 0.29 \tag{9}$$

where T is the temperature in Kelvin. Wien's law states that the wavelength of maximum emission from a blackbody is inversely proportional to its temperature. This makes sense; shorter wavelength light corresponds to higher energy photons, which you would expect from a higher temperature object.

Stefan-Botzmann's Law

It was mentioned at the beginning that the quantity of radiation emitted by a body depends on its temperature. In fact, the total radiation emitted by a body increases very rapidly as the temperature is raised.

According to Stefan-Boltzmann, ***the rate of emission of radiation from a black body is directly proportional to the fourth power of its absolute temperature***. So the rate at which energy leaves the black body is

$$\Re = \frac{dQ}{dt} = \sigma T^4 \qquad (10)$$

where $\Re$ is the radiation power per unit area and, σ is a universal constant called the Stefan-Boltzmann constant which has the value $\sigma = 5.67 \times 10^{-8}\,\text{watt/m}^2\text{K}^4$.

If the body is not perfect black and its emissivity is ε, then

$$\Re = \frac{dQ}{dt} = \varepsilon\sigma T^4 \qquad (11)$$

2. **Find the relation between the partition function Z and thermodynamic functions U, S, F and P.**

------------------------------------- **Solution** -------------------------------------

(a) Relation between Z and U

Since

$$Z = \sum_i g_i e^{\varepsilon_i / KT}$$

differentiate Z with respect to T, holding V constant,

$$\left(\frac{\partial Z}{\partial T}\right)_V = \sum_i g_i \left(\frac{\varepsilon_i}{KT^2}\right) e^{\varepsilon_i / KT}$$

$$= \frac{1}{KT^2} \sum_i \varepsilon_i g_i \; e^{\varepsilon_i / KT}$$

$$= \frac{1}{KT^2} \frac{\sum_i n_i \varepsilon_i}{\sum_i n_i} g_i \; e^{\varepsilon_i / KT}$$

$$= \frac{ZU}{NKT^2}$$

It follow that

$$U = NKT^2 \left(\frac{\partial \ln Z}{\partial T}\right)_V \qquad (8)$$

and U may be calculated once lnZ is known as a function of T and V.

(b) Relation between Z and S

The entropy S is related to the order or distribution of the particles, through the relation:

$$S = K \ln W$$

but

$$\ln W = -\sum_i n_i \ln \frac{n_i}{g_i} + N \ln N$$

Hence

$$S = K \ln W = K\left[-\sum_i n_i \ln \frac{n_i}{g_i} + N \ln N \right]$$

By using the relation

$$n_i = \frac{N}{Z} g_i e^{-\varepsilon_i / KT}$$

we have

$$\frac{n_i}{g_i} = \frac{N}{Z} e^{-\varepsilon_i / KT}$$

then

$$\begin{aligned} S &= K \ln W = K\left[-N \ln N + N \ln Z + \frac{U}{KT} + N \ln N \right] \\ &= NKT \ln Z + \frac{U}{T} \end{aligned} \tag{9}$$

and S may be calculated once lnZ is known as a function of T and V.

(c) Relation between Z and F

The property of the system is defined by its Helmholtz function F which is given by:

$$F = U - TS$$

This equation can be evaluated in terms of the partition function Z. By using the entropy S, Eq. (8), we get

$$F = U - T\left(NK \ln Z + \frac{U}{T} \right)$$

$$F = -NKT \ln Z \tag{10}$$

and F may be calculated once lnZ is known as a function of T and V.

(d) Relation between Z and P

Since

$$F = U - TS$$

then

$$dF = dU - TdS - SdT$$

From the first law of thermodynamics:

$$TdS - dU = PdV$$

By substituting

$$dF = -PdV - TdS$$

at constant temperature $SdT = 0$, then

$$P = -\left(\frac{\partial F}{\partial V}\right)_T$$

but

$$\frac{\partial F}{\partial V} = -NKT\frac{\partial \ln Z}{\partial V}$$

So

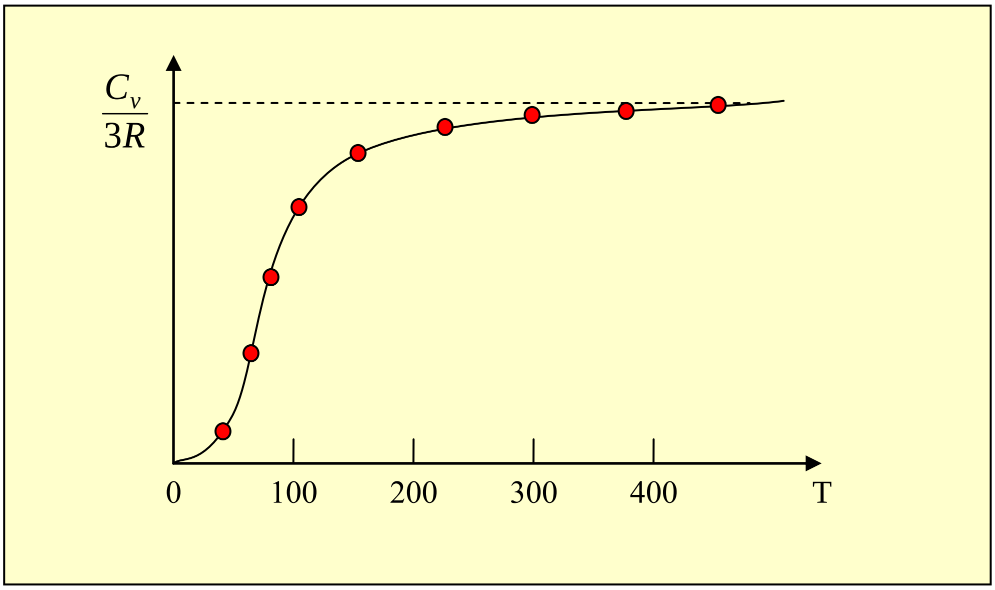
$$P = NKT\left(\frac{\partial \ln Z}{\partial V}\right)_T$$

and P may be calculated once lnZ is known as a function of T and V.

3. Discus the classical theory interpretation for Duliong-Petit law of specific heat

------------------------------------- **Solution** -------------------------------------

The specific heat depends on the temperature as in the figure. At high temperature the value of C_v is close to 3R



In classical theory the average energy is

$$\bar{\varepsilon} = KT \tag{1}$$

And the energy per mole is

$$U = 3N_A KT = 3RT \tag{2}$$

So the specific heat at constant volume is

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3R \tag{3}$$

This is in agreement with experiment at high temperature, but it fails completely at low temperatures.

4. Prove the following relation for the occupation number n_i due to Bose-Einstein Statistics $n_i = \frac{g_i}{e^{(\alpha+\varepsilon_i)/KT} - 1}$.

------------------------------------ **Solution** ------------------------------------

Let the number of allowed states associated with the energy ε_i be g_i. Let us first calculate the number of ways of putting n_1 particles of N particles in one box, then n_2 out of $N - n_1$ in second, and so on until we have exhausted all of the particles. The number of ways of choosing n_1 particles out of N particles is given by

$$W_1 = \frac{N!}{(N-n_1)!\, n_1!} \tag{1}$$

and the number of choosing n_2 out of $N - n_1$ is:

$$W_2 = \frac{(N-n_1)!}{(N-n_1-n_2)!\, n_2!} \tag{2}$$

and the number of ways of achieving this arrangement is

$$W = W_1 \cdot W_2 \ \cdots$$

$$= \frac{N!}{(N-n_1)!\, n_1!} \cdot \frac{(N-n_1)!}{(N-n_1-n_2)!\, n_2!} \ \cdots$$

$$= \frac{N!}{n_1!\, n_2! \ \cdots \ n_i!}$$

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i} \tag{3}$$

$$\ln W = \ln N! + \sum_i (n \ln g_i - n \ln n_i!)$$

$$= N \ln N + \sum_i (n \ln g_i - n \ln n_i)$$

To obtain the most probable distribution, we maximize Eq. (3) with $dN = 0$:

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - \frac{n_i}{n_i}) \delta n_i = 0$$

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - 1) \delta n_i = 0$$

but

$$\delta N = \sum_i \delta n_i = 0 \qquad (4)$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \qquad (5)$$

multiply Eq. (4) by $\alpha + 1$ and Eq. (5) bt $-B$ and add the resulting equations to each other:

$$\sum_i (\ln g_i - n \ln n_i + \alpha - \beta \varepsilon_i) \delta n_i = 0 \qquad (6)$$

Since n_i is vary independent,

$$\ln g_i - n \ln n_i + \alpha - \beta \varepsilon_i = 0$$

or

$$\ln \frac{g_i}{n_i} + \alpha - \beta \varepsilon_i = 0 \qquad (7)$$

Solving Eq. (7) for n_i gives

$$n_i = \frac{N}{Z} g_i e^{-\beta \varepsilon_i}$$