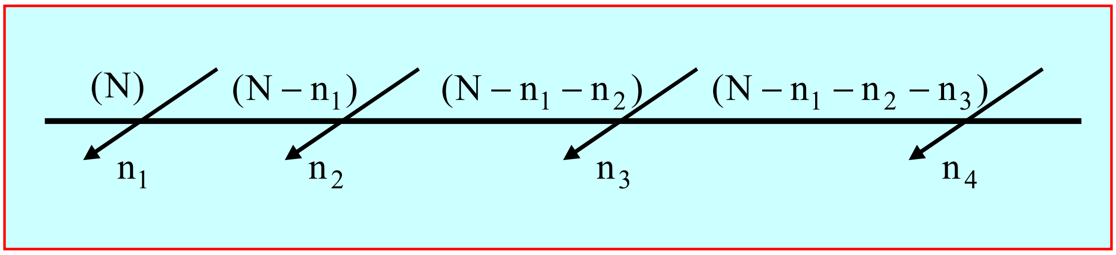
جامعة بنها — الفرقة الثالثة (فيزياء)
كلية العلوم — مادة (إحصائية)
دور يناير 2017 — نظام ساعات معتمدة — الزمن 3 ساعات

د./ صلاح عيد إبراهيم حمزة — تاريخ الامتحان: 31 /2016/12

1. Prove the following relation for the occupation number n_i due to Boltzmann distribution $n_i = \sum_i \frac{N}{Z} e^{-\beta\varepsilon}$

------------------------------ **Solution** --

Let the number of allowed states associated with the energy ε_i be g_i. Let us first calculate the number of ways of putting n_1 particles of N particles in one box, then n_2 out of $N - n_1$ in second, and so on until we have exhausted all of the particles. The number of ways of choosing n_1 particles out of N particles is given by



$$W_1 = \frac{N!}{(N - n_1)!\; n_1!} \tag{1}$$

and the number of choosing n_2 out of $N - n_1$ is:

$$W_2 = \frac{(N - n_1)!}{(N - n_1 - n_2)!\; n_2!} \tag{2}$$

and the number of ways of achieving this arrangement is

$$W = W_1 \cdot W_2 \ \cdots$$

$$= \frac{N!}{(N-n_1)!\ n_1!} \cdot \frac{(N-n_1)!}{(N-n_1-n_2)!\ n_2!} \ \cdots$$

$$= \frac{N!}{n_1!\ n_2! \ \ \cdots \ \ n_i!}$$

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i} \tag{3}$$

$$\ln W = \ln N! + \sum_i (n \ln g_i - n \ln n_i!)$$
$$= N \ln N + \sum_i (n \ln g_i - n \ln n_i)$$

To obtain the most probable distribution, we maximize Eq. (3) with $dN = 0$:

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - \frac{n_i}{n_i})\delta n_i = 0$$

$$\delta \ln W = \sum_i (\ln g_i - n \ln n_i - 1)\delta n_i = 0$$

but

$$\delta N = \sum_i \delta n_i = 0 \tag{4}$$

$$\delta U = \sum_i \varepsilon_i \delta n_i = 0 \tag{5}$$

multiply Eq. (4) by $\alpha + 1$ and Eq. (5) bt $-B$ and add the resulting equations to each other:

$$\sum_i (\ln g_i - n \ln n_i + \alpha - \beta\varepsilon_i)\delta n_i = 0 \tag{6}$$

Since n_i is vary independent,

$$\ln g_i - n \ln n_i + \alpha - \beta\varepsilon_i = 0$$

or

$$\ln \frac{g_i}{n_i} + \alpha - \beta\varepsilon_i = 0 \qquad (7)$$

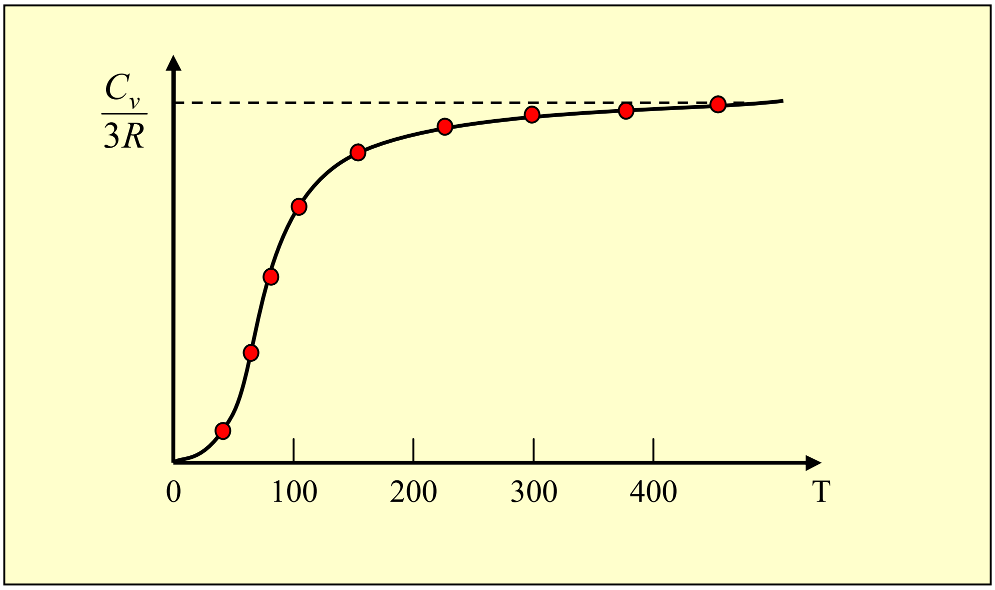
Solving Eq. (7) for n_i gives

$$n_i = \frac{N}{Z} g_i e^{-\beta\varepsilon_i}$$

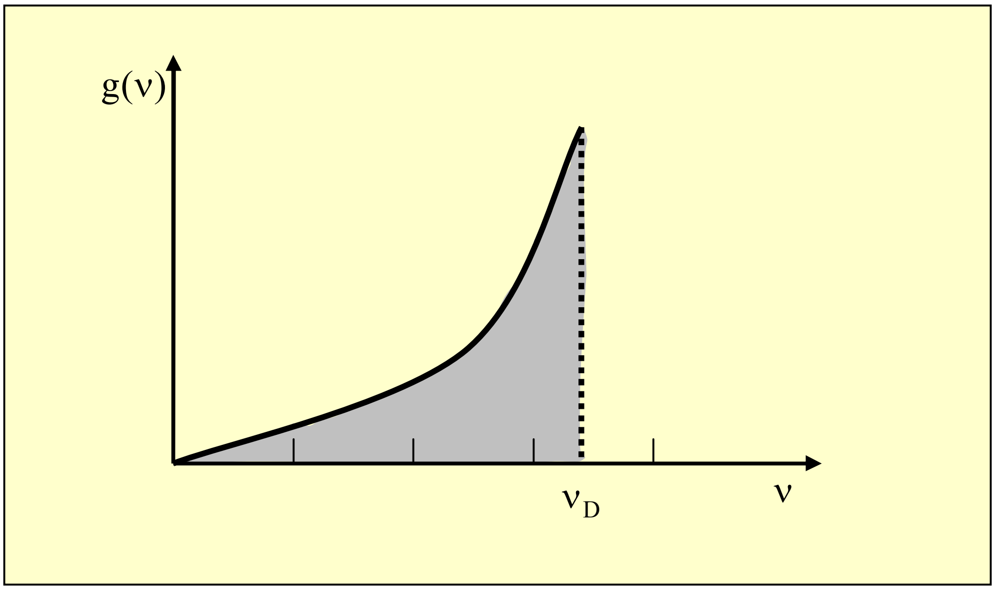
2. Debye treated with crystal as a continuous elastic medium and his expression of C_V is a good approximation to the Duling-Petit law. Discuss the previous paragraph.

------------------------------- **Solution** ---

The specific heat depends on the temperature as in the figure. At high temperature the value of C_v is close to 3R



In the Debye model, the frequency of the lattice vibration covrs a wide range of values. The lowest frequency in the Debye model is $\nu = 0$ and the highest allowed is ν_D such that the integral of $g(\nu)d\nu$ from 0 to ν_D equals 3N, see Fig. (2)



Thus

$$\int_0^{\nu_D} g(\nu)\,d\nu = 3N$$

By using the equation

$$g(\nu) = \frac{3V}{2\pi^2 c^3}\nu^2$$

We get

$$\frac{3V}{2\pi^2 c^3}\int_0^{\nu_D} \nu^2\,d\nu = 3N$$

$$\frac{3V}{2\pi^2 c^3}\frac{\nu_d^3}{3} = 3N$$

$$\nu_d^3 = \frac{6\pi^2 N c^3}{V}$$

Where ν_D is called Debye frequency. In terms of ν_D the function $g(\nu)$ is obtained as

$$g(\nu) = \frac{9N}{\nu_D^3}\nu^2 \qquad 0 \le \nu \le \nu_D$$

This summarizes the Debye theory of crystals.

3. Prove the following relation for the occupation number n_i due to Bose-Einstein Statistics $n_i = \dfrac{g_i}{e^{(\alpha+\varepsilon_i)/KT} - 1}$.

------------------------------- **Solution** -------------------------------

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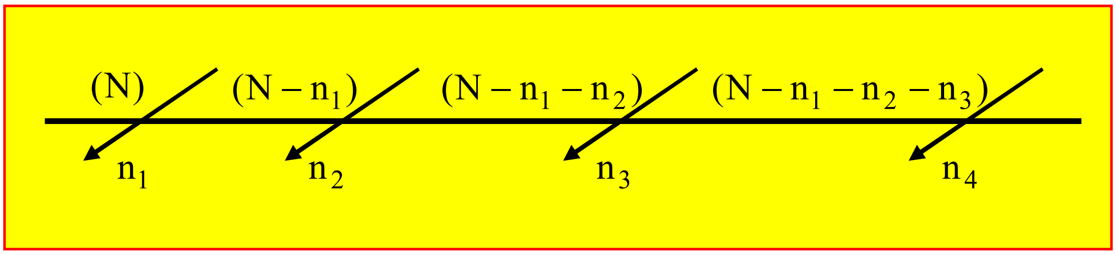

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and the number of ways of achieving this arrangement is

$$\begin{aligned} W &= W_1 \cdot W_2 \; \cdots \\ &= \frac{N!}{(N - n_1)!\; n_1!} \cdot \frac{(N - n_1)!}{(N - n_1 - n_2)!\; n_2!} \cdots \\ &= \frac{N!}{n_1!\; n_2! \; \cdots \; n_i!} \end{aligned}$$

$$W = N! \prod_i \frac{g_i^{n_i}}{n_i} \tag{3}$$

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Since n_i is vary independent,

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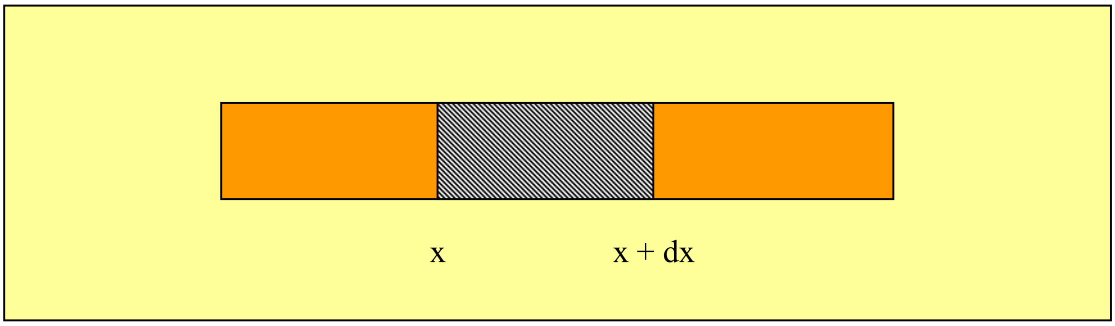
Solving Eq. (7) for n_i gives

$$n_i = \frac{N}{Z} g_i e^{-\beta\varepsilon_i}$$

4. Write a short note about the vibrational spectrum of crystals.

------------------------------------ **Solution** --------------------------------------

Let us examine the propagation of an elastic wave in a long bar. The wave equation in one dimension is



$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad (1)$$

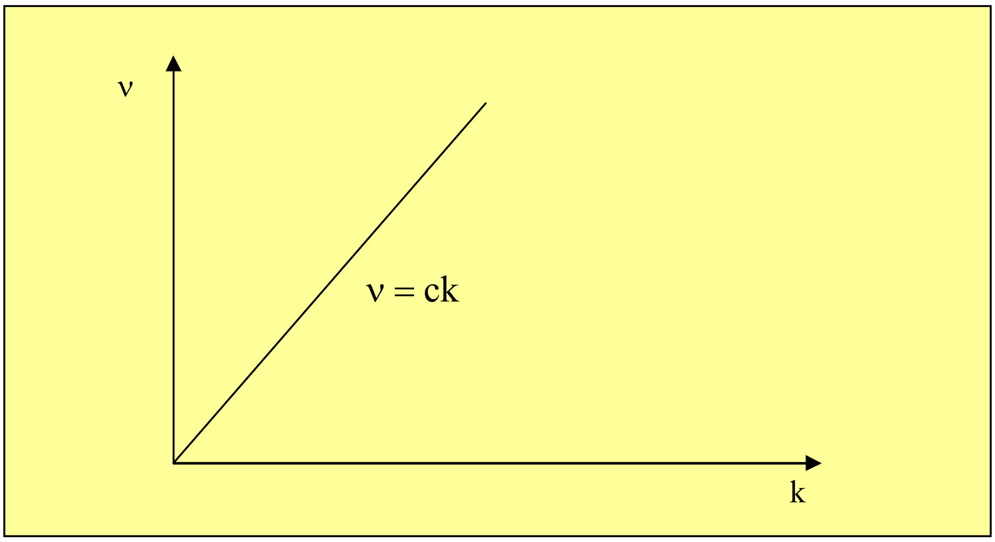
The solution of this equation is

$$\varphi = Ae^{i(kx - \nu t)} \qquad (2)$$

Substituting Eq. (2) in (1) leads to

$$\nu = ck \qquad (3)$$

The last equation is known as the dispersion relation which represents a straight line as in the figure

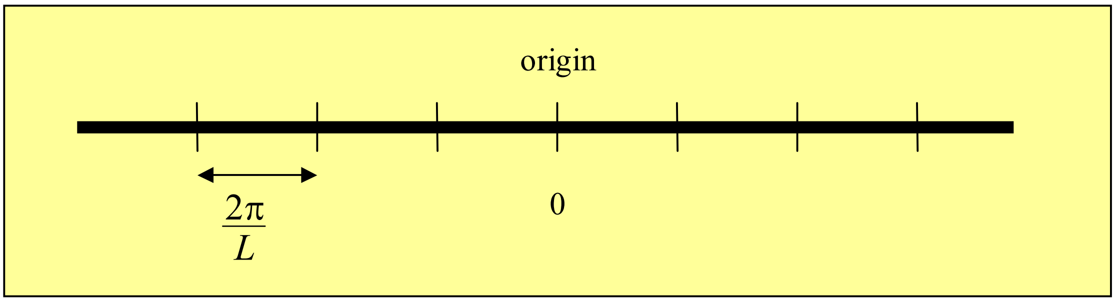


The boundary conditions require that

$$\varphi(0) = \varphi(L) \tag{4}$$

Substituting by Eq. (2) in (4) gives

$$k = n\frac{2\pi}{L}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{5}$$



The density of states is

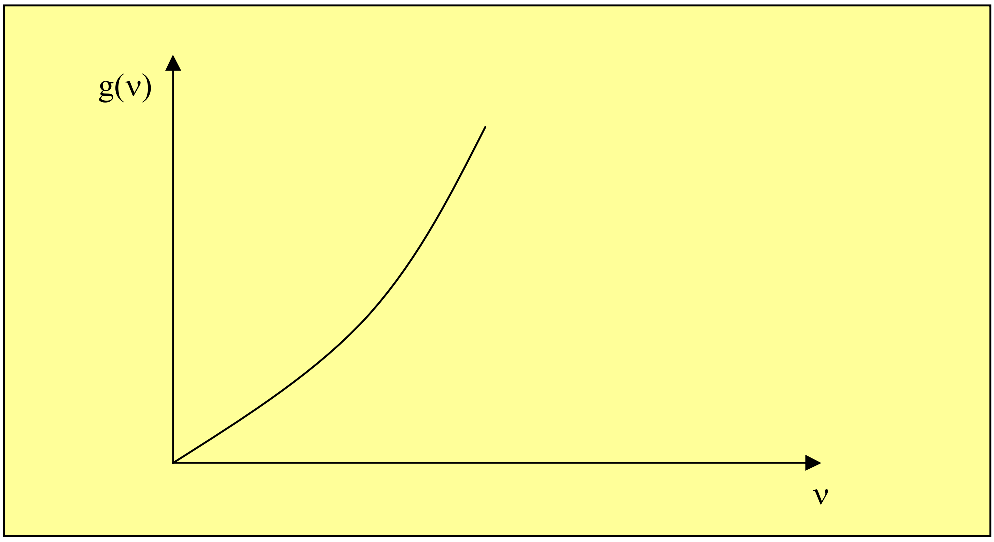
$$g(\nu)d\nu = \frac{L}{2\pi}dk \tag{6}$$

In one dimension

$$g(\nu) = \frac{L}{2\pi}\frac{1}{c}$$

In three dimension

$$g(\nu) = \frac{3V}{2\pi^2}\frac{\nu^2}{c^3} \tag{7}$$



Q5. (a) Differentiate between bosons and fermions.

----------------------------------- Solution --------------------------------------

fermions	Boson
Anti-Symmetric wave function	Symmetric wave function
Odd atoms - Protons - electrons	Even atoms - photons
Fermi Dirac statistics	Bose-Einstein statistics

Q5 (b) Discus in details the black body radiation phenomena

----------------------------------- Solution --------------------------------------

The Black body radiations can be considered as the photon gas. Photons are taken as bosons and they are obey BE statistics

$$n_i = \frac{g_i}{e^{\varepsilon_i/KT} - 1} \qquad (1)$$

We can write

$$dn = \frac{g(\nu)d\nu}{e^{h\nu/KT} - 1} \qquad (2)$$

The translational kinetic energy for a particle in a cubical box is

$$\varepsilon = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) \qquad (3)$$

In gama space

$$r^2 = n_x^2 + n_y^2 + n_z^2 \qquad (4)$$

$$r^2 = \frac{8mL^2}{h^2}\varepsilon \qquad (5)$$

So

$$gd\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon \qquad (6)$$

Since the photon has no rest mass, so we can write

$$g d\nu = \frac{4\pi V}{h^3} \frac{h^2 \nu^2}{c^2} \frac{h}{c} d\nu \tag{7}$$

The energy per unit volume, energy density, is

$$\rho d\nu = \frac{dn}{V} h\nu \tag{8}$$

So

$$\rho d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu / KT} - 1} d\nu \tag{9}$$

Which represents Planck's radiation law