جامعة بنـها - كلية العلوم – قسم الرياضيـات

المستوى الثالث(شعبة رياضيات)

الفصل الدراسي الأول

يوم الامتحان: السبت 2017/1/14 م

المادة : الأسس الرياضية لنظرية ميكانيكا الكم (M331)

أستاذ المادة : د . / خليل محمد خليل محمد

مدرس بقسم الرياضيات بكلية العلوم

صورة من الامتحان+ نموذج إجابته

Faculty of Science	Third level (Math.)	14 / 1 / 2017
Math. Department Benha	(**Quantum & Statistical) Mechanics**	Time: 2 hours

==

First: Mathematical Foundations of Quantum Theory (M331) questions

1.a	Find the adjoint operator $\hat{A}^{+}$ if $\hat{A}=\frac{d}{dx}$ defined on L_2 i.e. $\hat{A}\varphi(x)=\frac{d}{dx}\varphi(x)$ with the boundary condition $\varphi(\pm\infty)=0$. **(6 Marks)**
1.b	Show that: the eigenvalues of a unitary operator are complex numbers of unit modulus and its eigenvectors corresponding to unequal eigenvalues are mutually orthogonal? **(7 Marks)**
1.c	A particle of mass μ and energy E approaches a square potential barrier $U(x)=0,\ x<0$ and $U(x)=U_0,\ x\geq 0$ where $U_0>0$ from the left. Prove that the reflection coefficient R and the transmission coefficient T satisfy the relation $R+T=1$ if $E>U_0$. **(7 Marks)**
2.a	State the postulates of quantum mechanics. **(10 Marks)**
2.b	A particle whose Hamiltonian $\hat{H}$ has discrete spectrum E_n, where $\hat{H}\varphi_n(x)=E_n\varphi_n(x),\quad <\varphi_m(x)\vert\varphi_n(x)>=\delta_{mn},\quad E_n=(n+\frac{1}{2})\hbar\omega,\ n=0,1,2,....$ is described at $t=0$ by the normed state function $$\psi(x;0)=\sqrt{\frac{1}{5}}\varphi_0(x)+\sqrt{\frac{1}{2}}\varphi_2(x)+C_3\varphi_3(x).$$ (i) Determine the numerical value of C_3. (ii) Write out $\psi(x;t)$. (iii) What is $<E>_{t=0}$ and $<E>_{t=1}$. **(10 Marks)**

Look the Statistical Mechanics Exam

Dr. Khalil Mohamed

إجابة السؤال 1.a:

Proof: To obtain the adjoint operator, we take the inner product

$(\hat{A}\phi,\psi) = \int_{-\infty}^{\infty}(\hat{A}\phi(x))^{*}\psi(x)dx = \int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^{*}\psi(x)dx$ then by partial integration

let $u = \psi(x)$ and $dv = \frac{d}{dx}\phi(x)^{*}dx$ this leads to $du = d\psi(x)$, $v = \phi(x)^{*}$ then

$\int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^{*}\psi(x)dx = \psi(x)\phi(x)^{*}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi(x)^{*}(\frac{d}{dx}\psi(x))dx$. From the boundary condition

$\phi(\pm\infty) = 0$ then $\int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^{*}\psi(x)dx = 0 + \int_{-\infty}^{\infty}\phi(x)^{*}(-\frac{d}{dx}\psi(x))dx = (\phi,(-\frac{d}{dx})\psi) = (\phi,\hat{A}^{+}\psi)$

$\therefore\ \hat{A}^{+} = -\frac{d}{dx}.$

إجابة السؤال 1.b:

Proof:

where $\hat{U}\psi_i = \lambda_i\psi_i\ ;\ \psi_i \neq 0\ and\ \hat{U}\psi_j = \lambda_j\psi_j\ ;\ \psi_j \neq 0$ be a unitary operator. Let $\hat{U}$ Let

$\lambda_i \neq \lambda_j$ **f**$or\ i \neq j$ **.** Now

$(\hat{U}\psi_j,\hat{U}\psi_i) = \lambda_j^{*}\lambda_i(\psi_j,\psi_i)$ (i)

by definition

$(\hat{U}\psi_j,\hat{U}\psi_i) = (\psi_j,\psi_i)$ (ii)

from (i) and (ii)

$(1-\lambda_i\lambda_j^{*})(\psi_j,\psi_i) = 0$ (iii)

then $i = j\ in\ (iii)$ if $(1-\lambda_i\lambda_i^{*})(\psi_i,\psi_i) = 0$

Since $(\psi_i,\psi_i) \neq 0$ then $(1-\lambda_i\lambda_i^{*}) = 0 \Rightarrow |\lambda_i|^2 = 1 \quad \therefore |\lambda_i| = 1$

Thus the eigenvalues are complex numbers of unit modulus.

by assumption $\lambda_i \neq \lambda_j$ then $i \neq j\ in\ (iii)$ if

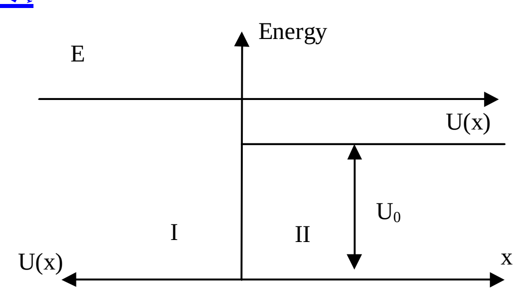
Since $(\psi_i,\psi_i) \neq 0$ then $(1-\lambda_i\lambda_i^{*}) = 0 \Rightarrow |\lambda_i|^2 = 1 \quad \therefore |\lambda_i| = 1$

$\lambda_i \neq \lambda_j \Rightarrow \lambda_i\lambda_j^{*} \neq \lambda_j\lambda_j^{*} = |\lambda_j|^2 = 1\ then\ \lambda_i\lambda_j^{*} \neq 1$

from (iii) $\Rightarrow (\psi_j,\psi_i) = 0$

Therefore, eigenvectors corresponding to unequal eigenvalues are mutually orthogonal.

إجابة السؤال 1.c:



The energy equation or Shrodinger equation may be written as:

$$[\frac{d^2}{dx^2}+\frac{2\mu}{\hbar}(E-U(x))]\psi_E=0 \qquad (1)$$

According the potential regions, equation (1) becomes

$$\left.\begin{array}{lll} \psi_I''+k_0^2\psi_I=0, & k_0=\frac{1}{\hbar}\sqrt{2\mu E}, & x<0 \\ \psi_{II}''+k^2\psi_{II}=0, & k=\frac{1}{\hbar}\sqrt{2\mu(E-U_0)}, & x\ge 0 \end{array}\right\} \qquad (2)$$

The general solution of system (2) is

$$\left.\begin{array}{ll} \psi_I(x)=A\exp(ik_0x)+B\exp(ik_0x), & x<0 \\ \psi_{II}(x)=C\exp(ikx)+D\exp(-ikx), & x\ge 0 \end{array}\right\} \qquad (3)$$

Since the beam incident from the left, then D must vanish.

Continuity Conditions

$$\left.\begin{array}{l} \psi_I(0)=\psi_{II}(0)\Rightarrow A+B=C \\ \psi_I''(0)=\psi_{II}''(0)\Rightarrow k_0A-k_0B=kC \end{array}\right\} \qquad (4)$$

From (4)

$B=(\frac{k_0-k}{k_0+k})A$ and $C=(\frac{2k_0}{k_0+k})A$ Thus (3) becomes

$$\psi_E(x)=A\begin{cases} \exp(ik_0x)+(\frac{k_0-k}{k_0+k})\exp(-ik_0x) & x<0 \\ (\frac{2k_0}{k_0+k})\exp(ikx) & x\ge 0 \end{cases} \qquad (5)$$

Since the reflection coefficient R and the transmission T coefficient satisfy the relations $R=\left|\frac{j_{ref}}{j_{inc}}\right|$ and $T=\left|\frac{j_{tran}}{j_{inc}}\right|$ respectively, where $j(x,t)=(\frac{\hbar}{\mu})\operatorname{Im}(\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x})$.

Now, let us calculate R and T. From the values for A and B with equation (5), one gets

$j_{inc}=\frac{\hbar k_0}{\mu}|A|^2,\quad J_{ref}=-\frac{\hbar k_0}{\mu}|B|^2,\quad J_{tran}=\frac{\hbar k}{\mu}|C|^2\quad \therefore R=\left|\frac{B}{A}\right|^2=(\frac{k_0-k}{k_0+k})^2$

And $T=\frac{k}{k_0}\left|\frac{C}{A}\right|^2=\frac{4k_0k}{(k_0+k)^2}$

Thus

$$R+T=(\frac{k_0-k}{k_0+k})^2+\frac{4k_0k}{(k_0+k)^2}=\frac{k_0^2-2k_0k+k^2+4k_0k}{(k_0+k)^2}=\frac{k_0^2+2k_0k+k^2}{(k_0+k)^2}=1.$$

إجابة السؤال 2.a:

*The postulates of quantum mechanics are:

1)-Postulate I: Every physical state of a dynamical system (a particle) is represented at a given instant of time t by normed vector $|\psi\rangle_t$ in H. It is assumed that the state vector contains all the information which one can know about the state of the system at that instant of time. ψ and $e^{i\delta}\psi$ where $\delta^* = \delta$ represent the same physical state.

2)- Postulate II: To every dynamical variable A there corresponds an observable $\hat{A}$. The observable $\hat{x}$ and $\hat{p}$ must satisfy $[\hat{x}, \hat{p}] = i\hbar$. The rules for constructing the observable $\hat{A}$ corresponding to the dynamical variable A, in the $x-rep$ are as follows:

$$(i) x \to \hat{x} = x,\ t \to \hat{t} = t,\ p \to \hat{p} = -i\hbar \frac{d}{dx}$$

$$(ii) A(x, p, t) \to \hat{A} = A(x, -i\hbar \frac{d}{dx}, t).$$

3)- Postulate III: If a particle is in state $|\psi\rangle_t$, a measurement of a dynamical variable A which is represented by the observable $\hat{A}$ in the discrete spectrum $\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle,\ \langle\varphi_n|\varphi_n\rangle = \delta_{nm},\ \hat{1}_a = \sum_i |\varphi_i\rangle\langle\varphi_i|$ will

*yield one of the eigenvalues with probability a_i

$$\rho_\psi(a_i) = \frac{|\langle\varphi_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

** If the result of measurement is a_k, then the state of the system will change from $|\psi\rangle$ to $|\varphi_k\rangle$ as a result of measurement.

4)- Postulate IV: The state function $\psi(x,t)$ describing the state of a dynamical system obeys the following" Schrodinger time-dependent" equation $\hat{H}$ whose Hamiltonian is

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$$

إجابة السؤال 2.b:

$(i)\ \because \|\psi(x,0)\|^2 = 1,\ (normalized)$ $\quad \therefore \frac{1}{5} + \frac{1}{2} + |c_3|^2 = 1 \Rightarrow \frac{7}{10} + |c_3|^2 = 1$

$\therefore |c_3| = \sqrt{\frac{3}{10}}$

وهو المطلوب أولا

For finding $\psi(x,t)$ then

$$\because\ \psi(x,t)=\sum_n \exp(-iE_n t/\hbar)\phi_n(x) < \phi_n(x)\,|\,\psi(x,0)> \quad and$$

$$\because\ \psi(x,0)=\sqrt{\frac{1}{5}}\phi_0(x)+\sqrt{\frac{1}{2}}\phi_2(x)+\sqrt{\frac{3}{10}}\phi_3(x)$$

$$\therefore\ \psi(x,t)=\sqrt{\frac{1}{5}}\exp(-iE_0 t/\hbar)\ \phi_0(x)+\sqrt{\frac{1}{2}}\exp(-iE_2 t/\hbar)\phi_2(x)+\sqrt{\frac{3}{10}}\exp(-iE_3 t/\hbar)\phi_3(x) \quad (1)$$

$$\because\ E_n=(n+\frac{1}{2})\hbar\omega\ ,\ n=0,1,2,3,........$$

$$\therefore\ E_0=\frac{\hbar\omega}{2},\quad E_2=\frac{5\hbar\omega}{2},\quad E_3=\frac{7\hbar\omega}{2} \qquad (2)$$

From (1) and (2), we get

$$\therefore\ \psi(x,t)=\sqrt{\frac{1}{5}}\exp(-i\omega t/2)\ \phi_0(x)+\sqrt{\frac{1}{2}}\exp(-i5\omega t/2)\phi_2(x)+\sqrt{\frac{3}{10}}\exp(-i7\omega t/2)\phi_3(x)$$

وهو المطلوب ثانيا

For finding the probability for particles that exist in the n level

$$\rho_{\psi(x,0)}(E_n)=\left|<\phi_n(x)\,|\,\psi(x,0)\right|^2=\left|<\phi_n(x)\,|\sqrt{\frac{1}{5}}\phi_0(x)+\sqrt{\frac{1}{2}}\phi_2(x)+\sqrt{\frac{3}{10}}\phi_3(x)>\right|^2$$

$$where\ \ n=0,1,2,.....$$

$$\therefore\ \rho_{\psi(x,0)}(E_0)=\frac{1}{5},\qquad \rho_{\psi(x,0)}(E_2)=\frac{1}{2},\qquad \rho_{\psi(x,0)}(E_3)=\frac{3}{10}.$$

$$\because\ <E>_{t=0}=\sum_{n=0}^{\infty}\rho(E_n)E_n=\frac{1}{5}E_0+\frac{1}{2}E_2+\frac{3}{10}E_3$$

بالمثل

$$\rho_{\psi(x,t)}(E_n)=\left|<\phi_n(x)\,|\,\psi(x,t)\right|^2$$

$$=\left|<\phi_n(x)\,|\sqrt{\frac{1}{5}}\exp(-i\omega t/2)\ \phi_0(x)+\sqrt{\frac{1}{2}}\exp(-i5\omega t/2)\phi_2(x)+\sqrt{\frac{3}{10}}\exp(-i7\omega t/2)\phi_3(x)>\right|^2$$

$$where\ \ n=0,1,2,.....$$

$$\therefore\ \rho_{\psi(x,t)}(E_0)=\frac{1}{5},\qquad \rho_{\psi(x,t)}(E_2)=\frac{1}{2},\qquad \rho_{\psi(x,t)}(E_3)=\frac{3}{10}.$$

$$\because\ <E>_{t>0}=\sum_{n=0}^{\infty}\rho(E_n)E_n=\frac{1}{5}E_0+\frac{1}{2}E_2+\frac{3}{10}E_3$$

$$\therefore\ <E>_{t=0}=<E>_{t>0}$$

وهو المطلوب ثالثا

****