Answer the following questions :

Question(1):

(a) *Prove that every compact set is closed and bounded .* (5 marks)

(b) *Prove that the union of two compact sets is compact .* (5 marks)

(c) *Prove that every relatively compact set is bounded .* (5 marks)

(d) *Which of the following sets is compact and which is not ? and why ?* (5 marks)

$$A = \left\{ \frac{1}{n} : n \in N \right\} \qquad ; \qquad B = \{ b_1 , b_2 , b_3 , \ldots , b_n \}$$

Question(2):

(a) *Define a* G_δ *- set and* F_σ *- set . Prove that the intersection of the two* G_δ *- sets is a* G_δ *- set and the union of two* F_σ *- sets is an* F_σ *- set .* (12 marks)

(b) *Prove that any singleton set* $\{a\}$ *in a metric space is a closed* G_δ *- set .* (8 marks)

Question(3):

(a) *Show that the function* (7 marks)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

is not continuous at $x = 0$

(b) *Prove that every convergent sequence is bounded .* (7 marks)

(c) *Show that the sequence* $\left(\left(\frac{n+1}{n} \right)^n \right)_{n=1}^{\infty}$ *is a Cauchy sequence .* (6 marks)

Question(4):

(a) *Let* $B=\left\{\left(\frac{-n}{n+1}\right) \quad : \quad n\in N\right\}$. *Find* $Sup(B)$ *and* $Inf\ (B)$. *(7 marks)*

(b) *Prove that the countable union of countable sets is countable .* *(7 marks)*

(c) *Use* (b) *above to show that the set* *(6 marks)*

$$Q=\left\{\frac{n}{m} \quad ; \quad n\,,m\in Z\,,m\neq 0\right\}$$

Of all rational numbers is countable .

مع أطيب التمنيات بالنجاح

إجابة اختبار مادة تحليل حقيقى 413 ر المستوى الرابع شعبة رياضيات - كلية العلوم العام الدراسي 2019/2018
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Question(1):

(a) (i) Let M be a compact set and $x_\circ \in \overline{M} \Rightarrow \exists$ a seqc. $(x_n)_{n=1}^{\infty} \subset M : x_n \to x_\circ$ but M is compact and so $x_\circ \in M \Rightarrow \overline{M} \subseteq M \Rightarrow M$ is closed .

(ii) Let M be a compact set and suppose that M is unbounded . If $(x_n) \subset M$,then (x_n) is not bounded $\Rightarrow (x_n)$ is not convergent which contradicts the fact that M is compact . Therefore M must be bounded .

(b) Let A and B be two compact sets and $\eta_A = \{Q_\alpha \in \tau : \alpha \geq 1\}$ is an open cover of A and $\eta_B = \{Q_\alpha \in \tau : \alpha \geq 1\}$ be an open cover of B. Then $\exists$ two finite subcovers $\eta_A^* = \{Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_n}\}$ for A s.t. $A \subseteq \bigcup_{i=1}^{n} Q_{\alpha_i}$ (1)

And $\eta_B^* = \{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ for B s.t. $B \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$ (2)

(1),(2) $\Rightarrow A \cup B = \bigcup_{i=1}^{n} (Q_{\alpha_i} \cup G_{\alpha_i}) = \bigcup_{i=1}^{n} H_{\alpha_i}$, where $H_{\alpha_i} = Q_{\alpha_i} \cup G_{\alpha_i}$ is open . i.e. $A \cup B$ is compact .

(c) Let B be a relatively compact set.

Then $\overline{B}$ is compact $\Rightarrow \overline{B}$ is bounded $\Rightarrow \delta(\overline{B}) = \sup d(x,y) < \infty$, $x, y \in \overline{B}$

But since $B \subseteq \overline{B}$, it follows that $\delta(B) \leq \delta(\overline{B}) < \infty \Rightarrow B$ is bounded .

(d) (i) The set $A = \left\{\frac{1}{n} : n \in N\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is not bounded and hence is not compact .

(ii) The set $B = \{b_1, b_2, b_3, \ldots, b_n\}$ is finite and hence is compact .

Question(2):

(a) A set G is called a G_δ- set if $G = \bigcap_{n=1}^{\infty} G_n$, G_n are open sets .

A set F is called a F_σ- set if $F = \bigcup_{n=1}^{\infty} F_n$, F_n are closed sets .

If G_1, G_2 are two G_δ- set , then $G_1 \cap G_2 = \left(\bigcap_{n=1}^{\infty} G_n\right) \cap \left(\bigcap_{n=1}^{\infty} Q_n\right) = \bigcap_{n=1}^{\infty} (G_n \cap Q_n) = \bigcap_{n=1}^{\infty} H_n$, $H_n = G_n \cap Q_n$ are open sets .

$\therefore G_1 \cap G_2$ is a G_δ- set . Similarly , for F_σ- set .

(b) $\{a\}=\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n},a+\frac{1}{n}\right)$ *is a* G_δ *- set which is closed as any singleton set in a m.sp. is closed .*

Question(3):

(a) *If we take* $(x_n)_{n=1}^{\infty}=\left(\frac{1}{2n\pi}\right)$ *and* $(x_n^*)_{n=1}^{\infty}=\left(\frac{1}{2n\pi+\frac{\pi}{2}}\right)$, *then* $x_n\to 0$ *and* $x_n^{*/}\to 0$ *but*

$f(x_n)=\sin(2n\pi)=0\to 0$ *and* $f(x_n^*)=\sin\left(2n\pi+\frac{\pi}{2}\right)=\cos(2n\pi)=1\to 1$

$\therefore f(x)$ *is not continuous at* $x=0$.

(b) $T:l^2\to l^2:T(x_1,x_2,\ldots,x_n,x_{n+1},\ldots)=(0,0,\ldots,0,x_{n+1},\ldots)\Rightarrow\|T(x)\|_2=\left(\sum_{i=1}^{\infty}|x_i|^2\right)^{1/2}=\|x\|_2\Rightarrow T$

is Let $x_n\to x_\circ\Rightarrow|x_n-x_\circ|<\varepsilon\ \forall\ n\ge n_\circ$

$\therefore|x_n|=|x_n-x_\circ+x_\circ|\le|x_n-x_\circ|+|x_\circ|<\varepsilon+|x_\circ|,\ n\ge n_\circ$

If we take $K=\max\{|x_1|,|x_1|,|x_1|,\varepsilon+|x_\circ|\}$, *then* $|x_n|<K\ \forall\ n\ge 1\Rightarrow(x_n)$ *is bounded .*

(c) *Since* $\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n\to e$ *as* $n\to\infty\Rightarrow\left(\left(\frac{n+1}{n}\right)^n\right)_{n=1}^{\infty}$ *is a Cauchy seqc.*

Question(4):

(a) $B=\left\{-\frac{1}{2},-\frac{2}{3},-\frac{3}{4},\cdots\to -1\right\}$

$\therefore Inf(B)=-1\quad,\quad \sup(B)=\max(B)=-1/2$ *and* $\min(B)$ *does not exist .*

(b) *Let* $\eta=\{E_1,E_2,E_3,\ldots\}$ *be a family of countable sets , where*

$E_1=\{e_{11},e_{12},e_{13},\ldots\}$ *and* $E_2=\{e_{21},e_{22},e_{23},\ldots\}$ $,\ldots,E_n=\{e_{n1},e_{n2},e_{n3},\ldots\}\ldots\Rightarrow\bigcup_{n=1}^{\infty}E_n=\{e_{ij}:e_{ij}\in N\times N\}$

Which is countable as $N\times N$ *is countable .*

(c) $Q=Q^+\cup Q^-\cup\{0\}$, *where* $Q^+=\left\{\frac{n}{m}:n,m\in N\ ,\ m\neq 0\right\}$ *and* $f:Q^+\to N\times N:f\left(\frac{n}{m}\right)=(n,m)$ *is a bigection mapping i.e.* Q^+ *is countable . Also ,* Q^+ *is countable and* $\{0\}$ *is countable* $\overset{(b)}{\Rightarrow}Q$ *is countable .*